ON THE THEORY OF SUBMERGED LAMINAR JETS

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The theory of submerged laminar jets has been treated by many authors [1–5]. This paper presents several new results.

1. Formulation of the problem and the basic equation. The theory of submerged jet deals with the self-similar flows of a viscous compressible fluid which are due to a singularity, interpreted as a point source of momentum. These flows satisfy the Navier-Stokes equations

$$(\mathbf{V}, \nabla)\mathbf{V} = -\nabla\frac{p}{\rho} + \nu\Delta\mathbf{V}, \qquad \operatorname{div}\mathbf{V} = 0. \tag{1.1}$$

In a cylindrical system of coordinates (r, φ, z), the similar solutions of (1.1) must have the form

$$v_r = \frac{\nu}{r}U(\eta), \quad v_z = \frac{\nu}{r}W(\eta), \quad \frac{p}{\rho} = \frac{\nu^2}{r^2}\Pi(\eta), \quad \eta = \frac{z}{r}. \tag{1.2}$$

The functions v_r, v_z, p must satisfy the conditions

$$v_r = v_z = p = 0 \quad \text{at} \quad r = \infty \quad \text{or} \quad |z| = \infty,$$
$$v_r = 0, \quad v_z - \text{bounded}, \quad \partial v_z / \partial r = 0 \quad \text{at} \quad r = 0. \tag{1.3}$$

In addition to that, the kinematic momentum of the jet

$$I = 2\pi\int_0^\infty r v_z^2\, dr = 2\pi\nu^2\int_0^\infty \frac{W^2}{\eta}\, d\eta \tag{1.4}$$

is given.

Substitution of (1.2) in (1.1) yields the system

$$U'(W - \eta U) - U^2 = \eta\Pi' + 2\Pi + (1+\eta^2)U'' + 3\eta U', \tag{1.5}$$
$$W'(W - \eta U) - UW = -\Pi' + (1+\eta^2)W'' + 3\eta W' + W, \tag{1.6}$$
$$W' = \eta U'. \tag{1.7}$$

The conditions (1.3) yield

$$U = W = \Pi = 0 \quad \text{at} \quad \eta = \pm\infty. \tag{1.8}$$

Integrating (1.6) with respect to η, we obtain

$$\Pi = (1+\eta^2)W' + \eta W - W(W - \eta U) - {}^1/_2 C_1. \tag{1.9}$$

We introduce the new independent variable x and the function y, defined as

$$x = \eta(1+\eta^2)^{-1/2}, \qquad y = (W - \eta U)(1+\eta^2)^{-1/2}. \tag{1.10}$$

Substituting (1.9) in (1.5), we obtain

$$-(1-x^2)^2 y''' = C_1 - {}^1/_2(1-x^2)(y^2)'' - x(y^2)' + y^2. \tag{1.11}$$

Taking account of (1.7), we can express the functions W and U in terms of y

$$U = -(1-x^2)y' - xy, \qquad W = (1-x^2)^{1/2}(y - xy'), \tag{1.12}$$

where the primes denote differentiation with respect to x. Differentiating (1.11) and dividing by $1 - x^2$, we obtain

$$-(1-x^2)y^{\mathrm{IV}} + 4xy''' = -{}^1/_2(y^2)'''. \tag{1.13}$$

Integrating (1.13) once, we obtain

$$-(1-x^2)y''' + 2xy'' - 2y' = -{}^1/_2(y^2)'' - C_2. \tag{1.14}$$

Integrating (1.14) twice more, we obtain

$$2(1-x^2)y' + 4xy - y^2 = C_2x^2 + C_3x + C_4 = F(x). \tag{1.15}$$

Equation (1.15), which is a Riccati's equation, was first obtained by N. A. Slezkin [4]. Using (1.8), it can be shown that $C_1 = 0$ (an analogous proof can be found in [6]). To determine the constant C_2, we compare equations (1.11), (1.14), and (1.15) at $x = 0$. This yields

$$C_2 = C_4. \tag{1.16}$$

It should be noted that this relation between the coefficients was missed by N. A. Slezkin, who treated all coefficients on the right side of (1.15) as independent.

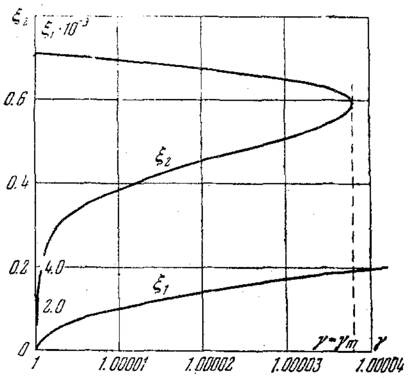

Fig. 1

2. Certain properties of Landau's solution. L. D. Landau's solution [1], which he interpreted as the discharge of a jet from an infinitely thin tube, is associated with the boundary conditions

$$y(\pm 1) = 0. \tag{2.1}$$

It can be easily seen that this requires

$$F(\pm 1) = 0. \tag{2.2}$$

Conditions (2.2), with equation (1.16), lead to the identities $C_2 = C_3 = C_4 = 0$. The solution of (1.15) is then

$$y = 2\frac{1-x^2}{\gamma - x}, \tag{2.3}$$

which has been obtained by Landau.

If one disregards relation (1.16), condition (2.2) is insufficient to determine the constants C_2, C_3, C_4, and the solution depends then not only on γ, but also on one other arbitrary parameter. Such a solution, which represents a whole class of jet flows, has been obtained by V. I. Yatseev [5]. The existence of relation (1.16) makes Landau's solution unique (V. I. Yatseev shows that of the whole class of solutions in his work only Landau's solution has physical meaning).

The constant γ is uniquely related to the momentum of the jet ($\gamma \to 1$ when $I \to \infty$).

As has been shown by Landau, in the case of a "strong" jet $\gamma = 1 + (1/2)\alpha^2$, $\alpha \ll 1$ his solution coincides with Schlichting's solution [2], obtained by the methods of boundary-layer theory.

$$v_z = \frac{8\nu}{z}\frac{\alpha^2}{(\alpha^2 + \xi^2)^2}, \qquad \alpha^2 = \frac{64\pi\nu^2}{3I},$$
$$\xi = \frac{r}{z}. \tag{2.4}$$

Let us analyze Landau's solution. Substituting (2.3) in the second equation in (1.12), we obtain

$$v_z = \frac{2\nu}{\gamma z}\frac{1}{\sqrt{1+\xi^2}}\left[1 + \frac{\gamma^2 - 1}{(\gamma\sqrt{1+\xi^2} - 1)^2}\right]. \tag{2.5}$$

Let us regard the variable z in (2.5) as constant and determine those values of r for which

$$\frac{\partial^2 v_z}{\partial r^2} = \frac{1}{z^2}\frac{\partial^2 v_z}{\partial \xi^2} = 0. \qquad (2.6)$$

We introduce the new variable $t = \sqrt{1+\xi^2}$ and the notation $m = \gamma t$ and $n = \gamma^{-2}$. Equation (2.6) becomes then

$$2m^5(m-4)n^2 + m^2(9m^2+4m-1)n - 3(5m^2-4m+1) = 0. \qquad (2.7)$$

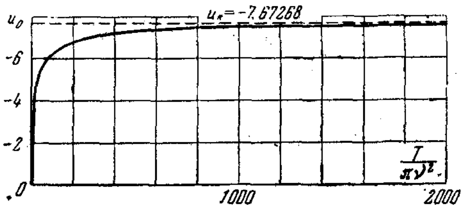

Fig. 2

Equation (2.7) determines a function n(m), which has two branches, corresponding to the two roots of the equation. Since $\gamma > 1$ and $t > 1$, it is interesting to consider the following range of the variables m and n: $1 \le m \le \infty$; $0 \le n \le 1$. One branch of n(m) is a monotonic function, which decreases from 1 to 0 as m increases from 1 to ∞. The other branch is not larger than n = 1 only in the range $1 \le m \le \sqrt{3/2}$. Since the values of n are very close to 1, it is convenient to introduce the variable $s = 1 - n$. Equation (2.7) then becomes

$$2m^5(m-4)s^2 - m^2(m-1)(4m^3-12m^2-3m+1)s + (m-1)^4(2m-3) = 0.$$

In the range $1 \le m \le \sqrt{3/2}$ the roots of this equation can be closely approximated by

$$s_1 = \frac{(m-1)(4m^3-12m^2-3m+1)}{3m^3(m-4)},$$
$$s_2 = \frac{(m-1)^3(2m^2-3)}{m^2(4m^3-12m^2-3m+1)}.$$

Knowing the functions n_1 and n_2, one can construct the functions $\xi_1(\gamma)$ and $\xi_2(\gamma)$, which are shown in Fig. 1. Fig. 1 shows that for $\gamma < \gamma_m$ there are three values of ξ for every γ, and for $\gamma > \gamma_m$ there is only one value. The value γ_m can be calculated by finding the maximum of $s_2(m)$. This yields $\gamma_m = 1.000038$. Thus, for $\gamma > \gamma_m$ the velocity profile v_z has one point of inflexion (whose existence is due to the boundary conditions), whereas for $\gamma < \gamma_m$ there appear two more points of inflexion. This, according to the well known Rayleigh's theorem, is a necessary condition for the absolute instability of the flow, as for $\gamma < \gamma_m$ one can expect the jet to become turbulent. We introduce the Reynolds number Re for a circular jet of radius a with a momentum equal to that of the point source:

$$I = 2\pi\int_0^a V_0^2 r\,dr = \pi V_0^2 a^2, \quad \mathrm{Re} = \frac{2}{\nu}\left(\frac{I}{\pi}\right)^{1/2} = 8\left[\frac{2}{2(\gamma-1)}\right]^{1/2}.$$

For $\gamma = \gamma_m$ we have Re = 1060.

3. On the ejection capacity of the jet. The ejection capacity of the jet can be characterized by the flow rate

$$G = 2\pi\int_0^\infty r v_z\,dr. \qquad (3.1)$$

Substituting (2.4) in (3.1), we obtain Schlichting's formula

$$G = 8\pi\nu z. \qquad (3.2)$$

Let us calculate G in its general form. The stream function is

$$\psi = 2\pi\int_0^r r v_z\,dr = 2\pi\nu\int_0^r W\,dr = 2\pi\nu\sqrt{r^2+z^2}\,y = 2\pi\nu z\left(\frac{W}{\eta} - U\right).$$

Consequently

$$G = 2\pi\nu z\lim_{\eta\to 0}\left(\frac{W}{\eta} - U\right). \qquad (3.3)$$

Hence, $G = \infty$ if $W(0) \ne 0$. In particular, $G = \infty$ for Landau's jet, which can also be obtained directly.

Thus, despite the fact that Schlichting's solution (2.4) coincides with Landau's exact solution for a "strong" jet near the axis, they give completely different values of G. Therefore, equation (3.2), which has been obtained by the unjustified application of (2.4) to the whole infinite flow field, is doubtful.

Consider the problem of finding a jet flow with a finite flow rate G. We have the necessary condition

$$W(0) = 0. \qquad (3.4)$$

Assuming in (1.10) $\eta = 0$, we find $y(0) = 0$. The symmetry of the conditions with respect to the z = 0 plane allows us to consider the problem in the region $z \ge 0$ only; therefore we use only one of the conditions (2.1): $y(1) = 0$. From (1.12) we find

$$y'(0) = -U(0) = -U_0.$$

Assuming x = 0 in (1.15), we find, taking into account (1.16),

$$C_2 = C_4 = 2y'(0) = -2U_0.$$

Using the condition F(1) = 0, we obtain $C_3 = 4U_0$ and

$$F(x) = -2U_0(1-x^2). \qquad (3.5)$$

The solution of (1.15) is in this case

$$y = 2U_0(1-x)\frac{\mathrm{th}[\chi\ln(1+x)]}{\mathrm{th}[\chi\ln(1+x)] - 2\chi} \qquad (3.6.1)$$

$$\text{for } U_0 < -\frac{1}{2} \quad (\chi = [{}^1\!/_2(U_0 + {}^1\!/_2)]^{1/2}),$$

$$y = -2U_0(1-x)\frac{(1+x)^{2\chi}-1}{(2\chi-1)(1+x)^{2\chi}+(2\chi+1)} \qquad (3.6.2)$$

$$\text{for } U_0 > -\frac{1}{2},$$

$$y = (1-x)\frac{\ln(1+x)}{2-\ln(1+x)} \quad \text{for } U_0 = -\frac{1}{2}. \qquad (3.6.3)$$

From (3.6) it can be seen that if one regards this flow as a jet discharging from an opening in a plane wall and imposes the no-slip condition U(0) = 0, the problem does not have a solution different from the trivial solution U = W = 0. However, there are three possible physical interpretations of the solution.

(a) One can regard the solution as representing a jet discharging from an opening in a wall which is impermeable but "perfectly smooth," so that the no-slip condition is inapplicable. Note that Schlichting's solution, valid over the whole half-space $z \ge 0$, corresponds to this case ($W(0) = 0$, $U(0) \ne 0$).

(b) One can regard the flow as produced by a dipole jet located at the origin, which consists of two equal jets flowing in opposite directions.

(c) The flow can be interpreted (for U(0) > 0) as radial discharge from a thin tube which lies on the z axis. As can be seen from (3.6.1), flows (a) and (b) are possible only for $U_0 > U_*$, since the function y has poles at $U_0 < U_* = -7.67268$.

The parameter U_0 is related to the momentum of the jet, and $U_0 \to 7.67268$ when $I \to \infty$. The relation between I and U_0 in the general case can be calculated numerically and is shown in Fig. 2. It can be seen that for sufficiently high values of the momentum the parameter U_0 is only weakly dependent on the momentum.

Let us find this relation for the case of the approximate solution for the "strong" jet. We shall show that in this case (3.6) coincides with Schlichting's solution near the axis. Let us rewrite (3.6.1) in the form

$$y = (1+b^2)(1-x)\frac{\mathrm{tg}[{}^1\!/_2 b\ln(1+x)]}{b - \mathrm{tg}[{}^1\!/_2 b\ln(1+x)]}$$

$$(b = \sqrt{-2U_0-1}, \quad b_0 = \sqrt{-2U_*-1}).$$

Further, let us expand the function y in a series of 1 − x and

$b_0 - b$ and let us take the linear terms of this expansion. The result is

$$y = (1 + b_0^2)\frac{b_0(1-x)}{{}^1/_4 a_0 b_0(1-x) + k_0(b_0 - b)},$$
$$a_0 = 1 + b_0^2, \quad k_0 = \frac{a_0 \ln 2}{2} - 1.$$

In accordance with (1.2) and (1.12), the function v_z for a "strong" jet near its axis is

$$v_z \approx \frac{v}{z} b_0 k_0 a_0 (b_0 - b)\frac{x^2}{[{}^1/_4 a_0 b_0 (1-x) + k_0(b_0 - b)]^2} \approx$$
$$\approx \frac{v}{z}\frac{a_0 b_0}{k_0(b_0 - b)}\left[1 + \frac{a_0 b_0}{2k_0(b_0 - b)}\frac{\xi^2}{4}\right]^{-2}. \quad (3.7)$$

One can easily see that the velocity profile (3.7) coincides with Schlichting's solution (2.4). Substituting (3.7) in the formula for the momentum, we obtain the approximate relation between I and U_0

$$I = \frac{16\pi v^2}{3} U_* \frac{\sqrt{-2U_* - 1}}{(-U_* \ln 2 - 1)(\sqrt{-2U_* - 1} - \sqrt{-2U_0 - 1}}. \quad (3.8)$$

Equations (3.8) and (3.3) determine the relation between the flow rate in a strong jet and its momentum. For $1 \to \infty$ the flow rate is finite. Substituting in (3.3) $U_0 = U_* = 7.67268$ and taking account of the fact that for $\eta \to 0$ equation (1.7) yields $\lim W/\eta = W'(0) = 0$, we find $G = 15.345336\,\pi\nu z$. Thus, the value of the flow rate in the "strong" jet is nearly twice as big as that calculated by Schlichting.

In conclusion we note that although a real wall is not "smooth," one can nevertheless neglect the effect of the boundary layer at sufficiently large distance away from the wall and can use the above solution.

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